

Invariance principle for ‘push’ tagged particles for a Toom Interface

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Abstract

In many interacting particle systems, tagged particles move diffusively upon subtracting a drift. General techniques to prove such ‘invariance principles’ are available for reversible processes (Kipnis-Varadhan) and for non-reversible processes in dimension $d > 2$. The interest of our paper is that it considers a non-reversible one-dimensional process: the Toom model. The reason that we can prove the invariance principle is that in this model, push-tagged particles move manifestly slower than second-class particles.

1 Introduction

Let us introduce the Toom model. It plays on spin configurations $\sigma := (\sigma(x))_{x \in \mathbf{Z}} \in \Omega$ with $\Omega = \{-1, 1\}^{\mathbf{Z}}$, but it is good to think of the different values of $\sigma(x)$ as the site x being occupied by either $+$ or $-$ particles. Each ± 1 particle is equipped with an exponential rate λ_{\pm} clock. When the clock rings for a particle of sign η , the particle exchanges positions with the first particle to its right of opposite sign $-\eta$. Since that opposite sign particle can be arbitrarily far away, this process is of infinite range, it is not a Feller process. Here and below, we’ll refer to this process as $\sigma_t := (\sigma_t(x))_{x \in \mathbf{Z}}$. The Bernoulli measures Ber_p , where $p = \text{Ber}_p(\sigma(x) = +)$, are invariant, and, in fact, we have showed [4] that they are the only invariant measures satisfying certain regularity conditions. In what follows, we are always referring to these stationary processes.

In the above description, there is an obvious notion of ‘tagged particle’, but it is not this notion for which we can prove the invariance principle. Instead, we consider Push-tagged particles: Let’s focus on a single signed particle and suppose that the block of spins to its immediate right has the same sign as the particle. Then, rather than viewing the particle as jumping *over* its neighboring block to the right, we can view the particle as moving to its right one site. In doing so it *pushes* the entire right neighboring block of particles one site right as well.

It is clear that this dynamics leads to the same *unlabeled* particle system as was defined above. This description of the dynamics provided inspiration for the paper [2] in

which the authors discussed a model called Push-ASEP which has a integrable structure. Note that Push-ASEP is really just the "totally asymmetric" case of the present setup with $\lambda_+ = 1, \lambda_- = 0$, see also a generalization: q-pushASEP, in [3]

A convenient feature of this "pushing" description is that the dynamics preserve any total ordering of particles of the same type. That is, if we denote by $Y_t^{(j)}$ the position, at time t , of the particle which starts at j and if $x < y$, then on the event that $\sigma_0(x) = \sigma_0(y)$

$$Y^{(x)}(t) \leq Y^{(y)}(t) \text{ for all } t > 0.$$

This simple observation is important in our proofs. Our main result is

Theorem 1.1 (Functional CLT for Tagged Push Particles). *Fix λ_+, λ_- nonnegative not both zero, and $p \in (0, 1)$. Starting from the Bernoulli measure Ber_p conditioned on $\sigma_0(0) = \pm 1$ (i.e. fixing the sign of the push particle),*

$$\frac{Y_{nt}^{(0)} - v_{\pm}nt}{\sqrt{n}} \xrightarrow{d} \sqrt{D}B_t, \quad t \in [0, 1]$$

where B_t is standard Brownian motion, the convergence is in distribution on Skorohod space and the drift is given by

$$v_{\pm} = \lambda_{\pm} \left(\frac{1}{1-p} \right) - \lambda_{\mp} \left(\frac{1-p}{p^2} \right)$$

The diffusion constant is positive $D > 0$.

The technique yields at the same time invariance principles for additive functions like

$$X_t = \int_0^t ds \sigma_s(0).$$

and integrated currents, like the total number of \pm particles crossing a given edge. However, for the additive functionals, we do not prove positivity of the variance.

Let us conclude by reviewing some earlier results on functional CLTs for tagged particles in conservative particle systems. The classic paper by Kipnis-Varadhan, [6], implies CLTs for symmetric exclusion processes (except the nearest neighbor case) while [9] extends this to general zero drift jump kernels. Both results work in any dimension. For non-zero drift, there is a general approach [8] for asymmetric exclusion processes in dimension $d \geq 3$. In dimensions $d = 1, 2$, there is no general approach available and results can only be proved on a case by case basis using specific features of the underlying models. Moreover, results in this case seem to be few and far between see for example [5, 7]. Our result also uses specific properties of the model, in particular fast mixing exhibited via natural coupling and the order-preserving feature of push particles.

1.1 Preliminaries: the Dynamics on \mathbf{Z} .

We fix once and for all $\lambda_{\pm} > 0$ with $\lambda_+ + \lambda_- = 1$, which just sets the overall time scale. We consider a sequence of i.i.d. rate one Poisson point processes $(N_x(t))_{x \in \mathbf{Z}}$ associated with vertices $x \in \mathbf{Z}$. Besides these Poisson point processes, the sample space on which our processes are defined supports a two dimensional array of i.i.d. uniform $[0, 1]$ variables $(U_{x,j})_{x \in \mathbf{Z}, j \in \mathbf{N}}$. Let $(\Omega, \mathbb{P}; \mathcal{B}_{\Omega})$ denote a probability space which supports all these variables. Define the filtration of sigma algebras $(\mathcal{F}_t)_{t \in \mathbf{R}^+}$ on \mathcal{B}_{Ω} by

$$\mathcal{F}_t = \sigma(N_x(s) : s \leq t; U_{x,k} : k \leq N_x(t)).$$

Let us define $\Omega = \{-1, +1\}^{\mathbf{Z}}$ and equip Ω with its natural product topology and associated Borel sigma algebra \mathcal{B} . Let $\Sigma = \Omega \times \Omega$ and equip it with its natural product sigma algebra. Finally, let $I \subset \mathbf{R}_+$ be any closed interval and let $D_{\Omega}(I) = D(I \rightarrow \Omega)$ be the space of càdlàg functions from I to Ω . In case $I = [0, \infty)$ we simply denote this space by D . We equip $D_{\Omega}(I)$ with the Skorokhod topology and associated Borel sigma algebra, the latter being denoted $\mathcal{B}(D_{\Omega}([0, \tau]))$.

In general, given a pair of measurable spaces $(\mathbf{X}, \mathcal{F}); (\mathbf{Y}, \mathcal{G})$ and a family of random variables $(X_i)_{i \in I}$, we shall denote by $\mathcal{B}(X_i : i \in I)$ the sigma algebra generated by the X_i 's. Also, given a measure μ on $(\mathbf{X}, \mathcal{F})$, the Lebesgue space $L^q(\mathbf{X}, \mu)$, $q \geq 1$ will often be abbreviated $L^q(\mu)$, and even L^q when confusion is unlikely, with corresponding norm denoted by $\|\cdot\|_{L^q(\mu)}$.

As already remarked, the process is non-Feller and therefore cannot be defined in the standard way (see [4] for further discussion on this point). Nevertheless, in [4] we constructed the process with Ber_p initial conditions. For λ_{\pm} fixed, it is convenient to introduce the thinned Poisson processes $N_{x,\pm}(t)$ by the differentials $dN_{x,+}(t) = \mathbf{1}\{U_{x,N_x(t)} < \lambda_+\} dN_x(t)$ and $dN_{x,-}(t) = dN_x(t) - dN_{x,+}(t)$.

Theorem 1.2. *There is a $\mathbf{P}_{\text{Ber}_p}$ -a.s. defined random variable $F : \Sigma \rightarrow D$, i.e. a càdlàg process, such that if we denote the value of F at time t by σ_t*

1. (Stationarity) σ_t is Ber_p -distributed for any t .
2. (SDE is satisfied) The SDEs

$$\begin{aligned} \sigma_{t_2}(x) - \sigma_{t_1}(x) = & \sum_{\eta=\pm 1} -2\eta \int_{t_1}^{t_2} dt \chi_x^{\eta}(\sigma_{t-}) dN_{x,\eta}(t) \\ & + \sum_{\eta=\pm 1} 2\eta \int_{t_1}^{t_2} dt \sum_{y < x} \chi_{[y,x-1]}^{\eta}(\sigma_{t-}) \chi_x^{-\eta}(\sigma_{t-}) dN_{y,\eta}(t) \end{aligned} \quad (1)$$

are satisfied $\mathbf{P}_{\text{Ber}_p}$ -a.s. In particular, the right hand side is absolutely summable and that the equality (1) holds for any x and $t_1 < t_2$.

Proof. This is a restatement, in slightly different language, of Lemma 2.8 of [4]. The main point here is the language of stochastic differentials to describe the evolution of spin variables. \square

Remark 1.3. *The $(U_{x,k})_{x,k}$ here may seem obscure. Through these variables we can couple an arbitrary collection of Toom trajectories $(\sigma_t^j)_{j \in J}$ indexed by an at-most countably infinite index set J . This was used in a variety of ways in [4]. In particular, we recall their use in Theorem 1.2. The key, and most concrete, step of the proof of Theorem 1.2 was the fact that, for short times one can couple a sequence of finite systems σ_t^L with periodic boundary conditions (on $[-L, L]$ say) so that for each finite window $[-K, K]$ and all $t \in [0, \epsilon)$, $\lim_L \sigma_t^L$ exists \mathbf{P} -a.s. and is Ber_p distributed for all $t \in [0, \epsilon)$. We will need this fact in Section 5.2 to verify a time reversal identity between a Toom process moving to the right and a Toom process moving to the left.*

In this paper, we only consider couplings between σ_t^j 's whose initial distribution is Ber_p . Formally,

Definition 1.4. *Let $\{(\sigma_t^j) : j \in J\}$ be two or more Toom processes (not necessarily on the same subset of \mathbf{Z}) having respective initial distributions Ber_p . When we discuss a “coupling of the $\{(\sigma_t^j) : j \in J\}$ started from μ ” we mean the following: μ is assumed to be a measure on $\prod_{j \in J} \{\pm 1\}^{\mathbf{Z}}$ whose marginals are the Ber_p . The coupling is then the collection of the D -valued random variables σ^j given by*

$$\sigma^j = F(\eta^j, \omega) \quad \sigma^j : \left(\prod_{j \in J} \{\pm 1\}^{\mathbf{Z}} \right) \times \Omega \rightarrow D.$$

The existence of a coupling started from a given μ is immediate from the fact that F is a $\mathbf{P}_{\text{Ber}_p}$ -a.s. almost-surely defined function and each single-spin-configuration marginal of μ is Ber_p . The law of $\{(\sigma_t^j) : j \in J\}$ starting from an initial measure μ will be denoted by \mathbf{P}_μ .

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2 Statement of Results and the Key Lemma

Our method of proof is fairly flexible, applying in wider generality than indicated in Section 1 with little extra overhead. It is convenient for us to formalize the collection of

observables which satisfy functional CLTs. Let us consider processes $X(t)$ of the following form.

$$dX(s) = \langle g(\sigma_{s-}), dN(s) \rangle + f(\sigma_{s-})ds. \quad (2)$$

where we used the notation

$$\langle g(\sigma_{s-}), dN(s) \rangle = \sum_{\eta, x} g_{\eta, x}(\sigma_{s-}) dN_{\eta, x}(s)$$

and $f, g_{\eta, x}$ are measurable functions $\Omega \rightarrow \mathbf{R}$.

We need a few assumptions on $f, g_{\eta, x}$. The first one imposes some regularity, in particular implying that equation Eq. (2) is well-defined.

Assumption 2.1 (Finite L^q -norms). *For any $1 \leq q < \infty$,*

$$\|f\|_{L^q(\text{Ber}_p)} + \sum_{\eta, x} \|g_{\eta, x}\|_{L^q(\text{Ber}_p)} < \infty$$

This assumption implies that $X(t)$ has finite variation on any finite interval, cf. Theorem 1.2 **P** a.s. Furthermore, we shall assume that the functions $f, g_{x, \eta}$ are well-approximated by local functions. For $f \in L^1(\text{Ber}_p)$, we consider the conditional expectations

$$P_R f(\sigma) := \mathbf{E}_{\text{Ber}_p}[f \mid \sigma(x), x \in [-R, R]].$$

Our second assumption reads

Assumption 2.2 (Local approximation).

$$\|f - P_R f\|_{L^2(\text{Ber}_p)} + \sum_x \|g_{x, \eta} - 1_{|x| \leq R} P_R g_{x, \eta}\|_{L^2(\text{Ber}_p)} < C e^{-cR}$$

Note that Assumptions 2.1 and 2.2 imply the bound in Assumption 2.2 holds with L^2 replaced by $L^q, q \geq 1$. Finally, we give a condition which restricts $X(t)$ to be measurable w.r.t. the path σ_t , i.e. to not depend on arrivals of the processes $N_{x, \eta}$ that have no bearing on the path σ_t .

Assumption 2.3 (Path measurability). *For any x, η ,*

$$g_{x, \eta}(\sigma) = \chi(\sigma(x) = \eta) g_{x, \eta}(\sigma)$$

Let Y_t denote the position of a tagged $+$ -particle with initial position 0. The process (σ_t, Y_t) with state space $\{(\sigma, y \in \Omega \times \mathbf{Z} : \sigma(y) = 1)\}$ is Markovian. We denote by $\mathbf{P}_{\text{Ber}_p, y}$ the probability measure for this process where σ_t is started from the measure Ber_p conditioned on the presence of a $+$ -particle at y and $Y_0 = y$. If we denote the spatial shifts $\tau_y : \Omega \rightarrow \Omega$ by $(\tau_y \sigma)_x := \sigma_{y-x}$, this process descends via the map $(\sigma, y) \mapsto \tau_y \sigma$ to a Markov process

on $\{-1, 1\}^{\mathbf{Z} \setminus \{0\}}$ called the *environment seen from the (push) particle*. It is easy to check the following, which is crucial for some of our results.

Lemma 2.4. *The Bernoulli measures on $\{-1, 1\}^{\mathbf{Z} \setminus \{0\}}$ are stationary when we pass to the environment-seen-from-the-push-particle perspective.*

We will also consider CLTs for processes $X(t)$ defined by the equation

$$dX(s) = \langle \tau_{Y_{s-}} g(\sigma_{s-}), \tau_{Y_{s-}} dN(s) \rangle + \tau_{Y_{s-}} f(\sigma_{s-}) ds \quad (3)$$

$\tau_y dN_{x,\eta}(s) := dN_{y+x,\eta}(s)$ and

$$\tau_y \cdot g_{x,\eta}(\sigma) := g_{x,\eta}(\tau_{-y} \cdot \sigma).$$

where f, g satisfy the assumptions above.

In general, we will refer to processes $X(t)$ defined by Eq. (2) or Eq. (3) with f, g satisfying the three Assumptions above as *quasi-local processes*. If it is necessary to distinguish between Eq. (2) and Eq. (3), we will refer to the latter as 'quasi-local w.r.t. to the tagged particle' and the former as quasi-local w.r.t. to the origin. We will often drop the subscript $\mathbf{P}_{\text{Ber}_p}, \mathbf{P}_{\text{Ber}_p, y}$ from our expressions below when there is no danger of confusion. One exception to this is the exposition of Section 4, where we deal with a coupling process and various initial measures.

We are now ready to state the main result.

Theorem 2.5. *Any quasi-local process $X(t)$ as defined above satisfies a Brownian invariance principle, i.e. the sequence of processes*

$$\frac{1}{\sqrt{n}}(X(nt) - nt v_X), \quad 0 \leq t \leq 1,$$

(with drift $v_X := (1/nt)\mathbf{E}(X(nt)) < \infty$) converges weakly, as $n \rightarrow \infty$, to a multiple of Brownian motion, in the Skorohod topology.

We list three important examples of such processes $X(t)$:

Corollary 2.6. *In particular, the invariance principle holds for*

1. *Additive functionals*

$$X(t) = \int_0^t ds f(\sigma_s)$$

with f satisfying the localization assumption.

2. *Tagged push-particles*

$$X(t) = Y_t.$$

3. *Integrated η -particle currents from $(-\infty, x)$ to $[x, \infty)$:*

$$X(t) = \sum_{y < x} \int_0^t \chi_{[y, x-1]}^\eta(\sigma_{s-}) dN_{y, \eta}(s). \quad (4)$$

To establish this corollary, we should check that these processes are indeed quasi-local process in the sense outlined above. For the first and third example, this is obvious, so we only comment on the tagged particle, Y_t . Let us consider only the case of the $+$ -particle. In the other case, there is an analogous representation. Let

$$Q_{x,r}(y, \sigma) := \mathbf{1}\{y \in [x, x+r)\} \chi_{[x, x+r)}^+ \chi_{\{x+r\}}^-, \quad (5)$$

$$P_{x,r}(y, \sigma) := \mathbf{1}\{y = x+r\} \chi_{[x, x+r)}^- \chi_{\{x+r\}}^+, \quad (6)$$

with empty products are treated as 1. One can check that

$$Y_t = \sum_{x \in \mathbf{Z}} \sum_{r > 0} \int_0^t Q_{x,r}(Y_{s-}, \sigma_{s-}) dN_{+,x}(s) - \sum_{r > 0} \int_0^t r P_{x,r}(Y_{s-}, \sigma_{s-}) dN_{-,x}(s)$$

Note here that by Lemma 2.4 the drift of Y_t satisfies

$$\mathbf{E}_{\text{Ber}_p, 0}[Y_t] = v_Y t$$

with

$$v_Y := v_Y(\lambda_+, \lambda_-, p) = \lambda_+ \left(\frac{1}{1-p} \right) - \lambda_- \left(\frac{1-p}{p^2} \right).$$

This establishes our main result Theorem 1.1, except for the positivity of the diffusion constant, which is however clear from the representation given in Section 7, where manifestly $D_1 > 0$ and $D_2 \geq 0$.

2.1 The Key Lemma

The perspective we shall take in proving our results is that $dX(t)$ is a random signed measure on any finite interval $I \subset \mathbf{R}^+$. Indeed, any real function of bounded variation defines a finite signed Borel measure. As remarked above, X is indeed a.s. of bounded variation on finite intervals. The space of finite signed Borel measures over a compact set $\mathcal{X} \subset \mathbf{R}^d$, equipped with the total variation norm, is a Banach space that we denote by $\mathcal{M}(\mathcal{X})$. It is the dual of $\mathcal{C}_b(\mathcal{X})$, the bounded continuous functions on X with the supremum norm. In all what follows, we take \mathcal{X} some finite rectangle in \mathbf{R}^d . Adding some standard considerations on Skorohod topology, we then derive

Lemma 2.7. *Fix a finite interval I . On (Σ, \mathbf{P}) , we have almost surely defined random variables ν , taking values in $\mathcal{M}(I)$, and given by $\nu(dt) := dX(t)$.*

Whenever we consider expressions involving multiple quasi-local processes, we will index them as $X^{(i)}$, with corresponding integrands denoted by $f^{(i)}, g^{(i)}$. To alleviate possible confusion, let us explicitly remark here that we will never mix the two cases of quasi-local processes and quasi-local w.r.t. a tagged particle. Now, given a finite collection $(X^{(i)})_{i=1}^\ell$ of quasi-local processes, the (random) product measure $\prod_i dX^{(1)}(s_i)$ is defined on the hypercube $[0, L]^\ell$ and we will always restrict these measures to the open simplex

$$\mathcal{S}_\ell(L) = \{(t_1, \dots, t_\ell) \in [0, L]^\ell, t_j < t_{j+1}\},$$

Most of our bounds will be phrased in terms of the variation of such measures. In particular, the key technical lemma we shall prove in the paper is stated as follows. Let $T > 1, \kappa > 1$ and set, for $l = 1, \dots, \ell - 1$

$$E_l(T, \kappa) = \{(t_1, \dots, t_\ell) \in \mathcal{S}_\ell(L) : t_l - t_1 \leq T, t_{l+1} - t_l \geq T^\kappa\}. \quad (7)$$

Constants are allowed to depend on the processes $X^{(i)}$, in particular on the $f^{(i)}, g^{(i)}$, unless explicitly stated otherwise.

Lemma 2.8. *Let $(X^{(i)})_{i=1}^\ell$ be quasi-local processes. Let μ be the measure given by*

$$\mu(dt_{1,\dots,\ell}) := \mathbf{E} \left[dX^{(1)}(t_1) \cdots dX^{(\ell)}(t_\ell) \right] - \mathbf{E} \left[dX^{(1)}(t_1) \cdots dX^{(l)}(t_l) \right] \mathbf{E} \left[dX^{(l+1)}(t_{l+1}) \cdots dX^{(\ell)}(t_\ell) \right]. \quad (8)$$

Then its variation $|\mu|$ on $E_l(T, \kappa)$ satisfies the bound

$$|\mu|(E_l(T, \kappa)) \leq CL^{\ell-l+1} e^{-cT^{\frac{\kappa-1}{2}}}.$$

This lemma sets the stage for us to prove functional CLTs via the method of moments, see Section 7.

3 Bounds on iterated integrals and random measures

This section provides a-priori bounds on the total variation of measures of the form

$$\nu(dt_{1,\dots,\ell}) = dX^{(1)}(t_1) \cdots dX^{(\ell)}(t_\ell) \quad (9)$$

on $\mathcal{S}_\ell(L)$. For a quasi-local process X , we use X_R to denote the local approximation to X obtained by replacing f by $f_R := \mathbf{P}_R f$ and $g_{x,\eta}$ by $g_{R,x,\eta} := 1_{|x| \leq R} \mathbf{P}_R g_{x,\eta}$ (see Assumption 2.2).

Lemma 3.1 (A priori bounds). *Let ν be the measure defined in Eq. (9) and let ν_R be the same but with all $X^{(i)}$ replaced by $X_R^{(i)}$. Then, for any R ,*

$$\|\nu|([0, t]^\ell)\|_{L^q(\mathbf{P})} \leq C(1 + t^\ell), \quad \|\nu - \nu_R|([0, t]^\ell)\|_{L^q(\mathbf{P})} \leq C(1 + t^\ell) e^{-cR}$$

with C depending on f, g but not on t, R .

To prove this lemma, the basic strategy will be to bound

$$|\nu|(\mathcal{S}_\ell(L)) \leq \prod_{j=1}^{\ell} |\nu^{(j)}|([0, L])$$

where $\nu^{(j)}(dt) = dX^{(j)}(t)$. One can apply Hölder's inequality to the RHS to get

$$\mathbf{E}[\| |\nu|(\mathcal{S}_\ell(L)) \|_{L^q(\mathbf{P})}] \leq \prod_{j=1}^{\ell} \left\| |\nu^{(j)}|([0, L]) \right\|_{L^{q\ell}(\mathbf{P})}$$

Note that

$$|\nu^{(j)}|([0, L]) = \bar{X}^{(j)}(L) - \bar{X}^{(j)}(0)$$

where $\bar{X}^j(t)$ is obtained from $X^j(t)$ by replacing f, g with $|f|, |g|$ in the definition of the process X . Obviously, $\bar{X}^{(j)}(t)$ is a quasi-local process and hence our task reduces to proving bounds on L^k norms of

$$I_t := \int_0^t dX(s) = X(t) - X(0)$$

when X is quasi-local. The following bound is useful for large t .

Lemma 3.2. *For any k ,*

$$\mathbf{E}[I_t^k]^{1/k} \leq C + Ct(\mathbf{E}[|f^k|]^{1/k} + \sum_{x,\eta} \mathbf{E}[|g_{x,\eta}|^k])$$

for constants C independent of f, g .

Note that f, g and $\tau_{Y_t}f, \tau_{Y_t}g$ have the same distribution under $\mathbf{P}_{\text{Ber}_p,0}$ so the RHS plays a similar role for quasi-local processes and quasi-local centered at a tagged particle.

Proof. Without loss of generality, we may assume $f, g \geq 0$. Using the stochastic integral representation of I ,

$$dI_t^k = \sum_{l=1}^k C(l)(dX_t)^l I_{t-}^{k-l}$$

where $C(l)$ are combinatorial factors and

$$(dX_t)^l := \langle g^l, dN_t \rangle \text{ for } l > 1.$$

Taking expectations, we have

$$\mathbf{E}[dI_t^k] = \sum_{l=1}^k C(l) \mathbf{E}[(\delta_{l,1} f dt + \sum_{x,\eta} g_{x,\eta}^l dt) I_t^{k-l}]. \quad (10)$$

Applying Holder's inequality to each term, with $1/p(l) + 1/q(l) = 1$ and $(k-l)q(l) = k$, we get

$$\frac{d}{dt} \mathbf{E}[I_t^k] \leq C \mathbf{E}[|f|^k]^{1/k} (1 + \mathbf{E}[I_t^k])^{1-1/k} + C \sum_{l=1}^k \sum_{x,\eta} \mathbf{E}[|g_{x,\eta}|^k]^{1/k} (1 + \mathbf{E}[I_t^k])^{1-l/k}.$$

Multiplying both sides by $\frac{1}{k}(1 + \mathbf{E}[I_t^k])^{1/k-1}$ and using the fact that $(1 + \mathbf{E}[I_t^k])^{(1-l)/k} \leq 1$ leads to a differential inequality which can be integrated. The lemma follows. \square

For small t we have a complimentary bound.

Lemma 3.3.

$$\mathbf{E}[I_t^2] \leq C(z + z^2), \quad \text{with } z = t(\|f\|_{L^2} + \sum_{x,\eta} (\|g_{x,\eta}\|_{L^2} + \|g_{x,\eta}\|_{L^2}^2))$$

for C independent of f, g .

Proof. We use (10) for $k = 2$ and we integrate the differential inequalities in the two regimes $\mathbf{E}[I_t^2] \leq 1$, $\mathbf{E}[I_t^2] > 1$, leading to the bounds z^2, z , respectively. \square

Proof of Lemma 3.1. The first inequality is immediate from Lemma 3.2. The second follows from Lemma 3.3 as well by replacing X by $X - X_R$ (so that the corresponding f, g are small by Assumption 2.2. The fact that Lemma 3.3 deals only with L^2 -bounds is bypassed by estimating

$$\|\nu - \nu_R(I)\|_{L^q} \leq \|\nu - \nu_R(I)\|_{L^2} (\|\nu(I)\|_{L^{2q-2}}^{1-1/q} + \|\nu_R(I)\|_{L^{2q-2}}^{1-1/q})$$

for $I = [0, t]$. The second factor is then estimated by Lemma 3.2. \square

4 Motion of discrepancies

In this section, we deal throughout with processes taking values in Ω^2 , or $\Omega^2 \times \mathbf{Z}$ when also considering tagged particles. Pairs of spin configurations are denoted by $\sigma = (\sigma^1, \sigma^2)$ with $\sigma^i \in \Omega$. A site x where $\sigma^1(x) \neq \sigma^2(x)$ is said to host a 'discrepancy', and we say the discrepancy is of sign $+$ when $(\sigma^1(x), \sigma^2(x)) = (+, -)$ and it is of sign $-$ when $(\sigma^1(x), \sigma^2(x)) = (-, +)$. Let $D(\sigma)$ denote the position of the left most discrepancy of σ ,

i.e.

$$D(\sigma) := \inf\{x : \sigma^1(x) \neq \sigma^2(x)\}.$$

We will always consider initial measures on Ω^2 so that $D > -\infty$ almost surely. For $S \subset \mathbf{Z}$, let μ_S be the initial measure on Ω^2 defined by the following conditions:

$$\begin{aligned} &\sigma^1 \text{ and } \sigma^2 \text{ are Ber}_p \text{ distributed,} \\ &\text{For } x \in S^c, \sigma^1(x) = \sigma^2(x), \\ &\text{For } x \in S, \sigma^2 \text{ is independent of } \sigma^1. \end{aligned} \tag{11}$$

That is, the measure μ_S places possibly discrepancies in all $x \in S$. The coupling construction defines a dynamics on discrepancies. For example, let the configuration be σ^1 (above), σ^2 (below):

$$\begin{array}{cccccccccccc} + & + & + & - & + & + & + & + & + & + & + & - \\ + & + & + & - & + & - & + & + & + & + & + & - \\ \cdots & \cdots & \cdots & \cdots & x & y & z & \cdots & \cdots & \cdots & \cdots & w \end{array}$$

The discrepancy (of sign $+$) sits at y . If the first clock ring (locally) is at x , then the discrepancy will move to site w . If the first clock ring is at y , then it will move to z or w , depending on the relevant random variable U . Other clock rings do not move the discrepancy. In fact, it is guaranteed that a clock ring on the site of the discrepancy and a clock ring on the site left to it will move the discrepancy forward by at least one site. In case there is more than one discrepancy around, the picture is slightly more complicated. Discrepancies of type $+$ can annihilate with discrepancies of type $-$ (they cannot cross each other) and discrepancies of the same type can possibly cross. What the latter means (to have a crossing of discrepancies) is a matter of convention. We will never need such considerations, and don't sort this out. For us, it is important to realize (by inspection of possibilities) that 1) the motion of an isolated discrepancy is independent of the presence of other discrepancies as long as it does not collide with or cross (or is crossed by) any of them, and 2) for the leftmost discrepancy D , it is in any case true that a clock ring on or left to that discrepancy will move it by at least one site. This leads to an immediate proof of the following bound:

Proposition 4.1 (Linear Displacement of Minimal Discrepancy I). *There are constants $c, C > 0$ such that for any $x \in \mathbf{Z}$ and all $t > 0$,*

$$\mathbf{P}_{\mu_{[x, \infty)}}(D(\sigma_t) - x < ct) \leq Ce^{-ct}.$$

Whenever the tagged particle is involved, we need the following tweak of the above estimate, showing that discrepancies run away from particle with a positive relative speed. Its proof appears in the next subsection. We write $\mathbf{P}_{\mu_S, 0}$ for the coupled process started from the coupling measure μ_S conditioned on $\sigma_0^1(0) = +$. That is, the convention is that the tagged particle is placed in the first configuration σ^1 . Therefore, we take $Y_t = Y_t(\sigma^1)$.

Proposition 4.2 (Linear Displacement of Minimal Discrepancy II). *There are constants $c, C > 0$ such that for any $x > 0$ and all $t > 0$,*

$$\mathbf{P}_{\mu_{[x, \infty)}, 0}((D(\sigma_s) - x) - (Y_s - Y_0) < cs \text{ for some } s \geq t) \leq Ce^{-ct}.$$

and $D(\sigma_t) - Y_t > 0$ for all $t \geq 0$, with probability 1.

4.1 Tagged Particles

We prove here Proposition 4.2. Let (σ_0^1, σ_0^2) be two initial configurations with $\sigma_0^1(0) = +1$. The proof relies on the introduction of a pair of orderings associated to the particles of σ^1 . The first (resp. second) ordering labels the $+$ (resp. $-$) particles relative to one another. The orderings are defined at $t = 0$ and preserved in time according to the "push" dynamics. To order the $+$ -particles at $t = 0$ we use the notation i^+ with $i \in \mathbf{Z}$. We set $0^+ = Y_0 = 0$ and label the i 'th particle to the right or left of 0^+ by i^+ depending on whether i is respectively positive or negative. We shall denote by $Y_t^{i^+}$ the position in \mathbf{Z} at time t of the particle labeled by i^+ . An analogous ordering of the $-$ -particles is fixed once we declare 0^- to be the first particle left of 0 at $t = 0$.

Next, we define locations in these orderings for the discrepancies appearing in σ_0 . Recall that a discrepancy can be either of sign $+$ or of type $-$ and its sign is conserved throughout its evolution, though, as already remarked, opposite discrepancies can annihilate.

Suppose there is a \pm -discrepancy at x at $t = 0$. Assuming that it did not by the time $t > 0$, denote its location in \mathbf{Z} by \mathfrak{d}_t^x . We'll give the another 'location' of a $+$ -(resp. $-$ -)discrepancy by specifying the label i^+ (resp. i^-) of the $+$ -(resp. $-$ -)particle the discrepancy sits on. That is, we set $d_t^x = i^\pm$ where i is such that $\mathfrak{d}_t^x = Y_t^{i^\pm}$. As long as the discrepancy is isolated, it is easy to see that d_t^x either increases or stays constant when a clock ring affects the discrepancy. In fact, if the clock at \mathfrak{d}_t^x rings and the relevant U -variable dictates the \mp particle to move, then d_t^x is guaranteed to increase by at least one. This means that the increase of d_t^x may be stochastically bounded from below by a rate $\min(\lambda_+, \lambda_-)$ Poisson process.

There is ambiguity in this reasoning when other discrepancies touches are present unless we focus on (only) the leftmost discrepancy of type \pm . In that case, among all potential outcomes, the only one requiring further explanation is when the leftmost \pm -discrepancy annihilates with one of opposite type. In that, case one of the discrepancies to its right becomes the leftmost discrepancy (or it is assigned the value ∞ , if there is no other discrepancy of the same type). The foregoing discussion, with $D^\pm(\sigma_t)$ denoting the position, in the \pm ordering, of the leftmost discrepancy of type \pm , proves the following:

Lemma 4.3. *There exist $C, c > 0$ such that*

$$\mathbf{P}_{\mu_{[x, \infty)}}(D^\pm(\sigma_t) - D^\pm(\sigma_0) < ct) \leq Ce^{-ct}.$$

We can now proceed with

Proof of Proposition 4.2. As long as the left-most discrepancy D has sign $+$, the claim is easy: The position of the discrepancy in the $+$ ordering is linearly increasing by Lemma 4.3, whereas the position of the tagged particle in the $+$ ordering is constant. This also implies a linearly growing distance on the lattice. When the left-most discrepancy has sign $-$, it takes values in a different ordering than the tagged particle, so the above argument fails. However, since the tagged particle is to the left of all discrepancies, its motion in σ^1 and σ^2 is the same. Therefore, one may now reverse the roles of σ^1 and σ^2 , thus flipping the sign of the discrepancy so it takes values in the same ordering as the tagged particle. \square

4.2 Upper Bound on Speed of Discrepancies

Above, we have argued that discrepancies move at least linearly to the right/away from tagged particles. Now we provide upper bounds.

Lemma 4.4. *For any $t \geq 1, R \geq 0$, we have*

$$\mathbf{P}_{\mu_{\{x\}}}(\sigma_0^1(x) \neq \sigma_0^2(x), D(\sigma_t) - x \geq R) \leq Ce^{-c(R/t)^{1/2}}$$

The same reasoning can be used to prove bounds on the displacement of the tagged particle.

Lemma 4.5. *For any $t \geq 1, R \geq 0$, we have*

$$\mathbf{P}_{Ber_p,0}(|Y_s - Y_0| \geq R \text{ for some } s \geq t) \leq Ce^{-c(R/t)^{1/2}}$$

To prove these results we need an *a-priori* flux bound: Let us define the counting processes

$$J_x(t) := \sum_{\eta} \sum_{y < x} \int_0^t \chi_{[y, x-1]}^{-\eta}(\sigma_{s-}) \chi_y^{\eta}(\sigma_{s-}) dN_{y,\eta}(s) \quad (12)$$

This process records the total number of particles which jump from $(-\infty, x)$ to $[x, \infty)$ in the time interval $[0, t]$. The following bound was proved in [4], see Lemma 4.7.

Lemma 4.6 (*A-Priori Flux Bound*). *There are constants $C, \gamma > 0$, depending only on λ_{\pm}, p , such that*

$$\mathbf{E}_{Ber_p}[e^{\gamma J_x(t)/t}] < C, \quad \text{for any } t > 0$$

Proofs of Lemmas 4.4 and 4.5. For concreteness, we restrict attention to the proof of Lemma 4.4, the argument being similar in the remaining case. Let us denote by l_x the

number of spins to the left of x (including x) of the same sign as $\sigma(x)$. Similarly, r_x is the number of like spins to the right, starting at x . By definition $l_x, r_x \geq 1$.

There are two ways a discrepancy at x can move: The first way is that the exponential rate one clock on the vertex it occupies rings. The other way is if one of the exponential rate one clocks at $x - l_x + 1, \dots, x - 1$ rings. Hence the local rate of moves is bounded by l_{D_t} . If such a move occurs, the jump length is bounded by r_{D_t} . So, if we can bound the size of stretches of like spins that the discrepancy encounters, we can bound the speed of the discrepancy.

Let E be the event that a stretch of spins of length at least L occurs in the spatial interval $[x - R, x + R]$ in the time interval $[0, t]$, hence not necessarily only adjacent to D_s for $s \in [0, t]$. We will nevertheless find a good bound on E and then estimate the motion on E^c straightforwardly. The parameter L will be fixed, depending on R, t at the end of the proof. We divide $[0, t]$ into t intervals of length of order 1 and divide $[x - R, x + R]$ into blocks of length L . We enumerate the corresponding spacetime rectangles of $[0, t] \times [-R, R]$ by $(B_j)_{j=1}^J$, where $J = O(Rt)$. We write $B_j = [s_j, t_j] \times [a_j, b_j]$ and consider the events

$$E_j = \{[a_j, b_j] \text{ has a stretch of } L/2 \text{ like spins at some } t \in [s_j, t_j]\}.$$

$$F_j = \{[a_j, b_j] \text{ has a stretch of } L/4 \text{ like spins at } s_j\}.$$

In order for E_j to occur, either there must already be a stretch of length $L/4$ present at time s_j , i.e. F_j occurs, or at least $L/4$ particles must cross some vertex $x \in [a_j, b_j]$ in the (small) time interval $[s_j, t_j]$. Both of these possibilities are unlikely: A large deviation estimate for Ber_p yields

$$\mathbf{P}(F_j) \leq Ce^{-cL}.$$

and the flux bound Lemma 4.6 bounds the probability that $L/4$ particles crossed a vertex, i.e.

$$\mathbf{P}(E_j | F_j^c) \leq Ce^{-cL}.$$

Hence we conclude that $\mathbf{P}(E_j) \leq Ce^{-cL}$ and hence $\mathbf{P}(E) \leq C(tR)e^{-cL}$. It remains to estimate the speed of the discrepancy condition on E^c . As explained above, the distance traveled is now bounded above by $LN_t^{(L)}$ with $N_t^{(L)}$ a Poisson process with intensity L . Large deviation estimates yield that $\mathbf{P}(LN_t^{(L)} \geq R) \leq Ce^{-cR/L}$ provided that $R \geq CtL^2$. Collecting the estimates, we obtain

$$\mathbf{P}_{\mu_{\{x\}}}(\sigma_0^1(x) \neq \sigma_0^2(x), D(\sigma_t) - x \geq R) \leq C(tR)e^{-cL} + Ce^{-cR/L}, \quad \text{for } R \geq CtL^2$$

which is optimized to give a bound $Ce^{-c\sqrt{R/t}}$, provided $t \geq 1$. \square

4.3 Decay of Correlations

For a random measure $\mu = \mu(\sigma)$ we write $\tau_{-x}\mu(\sigma) = \mu(\tau_x\sigma)$, i.e. same convention as for number-valued random variables. Also, as a natural extension of our previous notation we will say $\mu \in \mathcal{B}(\sigma(x) : x \in A)$ if for every $f \in C_b(S_i)$, the variable $\int f d\mu \in \mathcal{B}(\sigma(x) : x \in A)$.

Lemma 4.7 (Exponential Decay of Correlations). *Let U, V be random measures on compacts S_1, S_2 , respectively such that $U \in \mathcal{B}(\sigma(x) : |x| \leq M)$ and $V \in \mathcal{B}(\sigma(x) : x \leq M)$. Then*

$$\begin{aligned} & |\mathbf{E}_{Ber_p,0}[\tau_{Y_t} V(\sigma_t) U(\sigma_0)] - \mathbf{E}_{Ber_p,0}[V(\sigma_0)] \mathbf{E}_{Ber_p,0}[U(\sigma_0)]| (S_1 \times S_2) \\ & \leq C \|U\|_{L^4(S_1)} \|V\|_{L^4(S_2)} e^{-c(t-M)} \end{aligned} \quad (13)$$

where all $\|\cdot\|(S)$ stand for the variation on S . The same bound holds if we replace $\tau_{-Y_t} V$ with V and $\mathbf{E}_{Ber_p,0}$ with \mathbf{E}_{Ber_p} (i.e. the case with no tagged particle).

The statement in the absence of a tagged particle is simpler to prove. In fact, a weaker version applying to functions rather than measures, appears already in [4]. Thus, we explicitly prove here only the decay of correlations in the presence of a tagged particle. There are some technical complications, mostly due to the fact that if one tries to couple two tagged particles in two different environments, they will not necessarily lie on the same vertex in \mathbf{Z} after all discrepancies move to the right of them. To circumvent this difficulty, the idea is to focus on a tagged particle that starts to the left of all discrepancies.

Proof of Lemma 4.7. Without loss, let V to be of zero mean. Let

$$-M'(\sigma_0) = \inf\{x < -M : \sigma_0(x) = +\},$$

that is, $-M'$ is the position of the rightmost $+$ -particle to the left of $-M$. We set

$$Z_t(\sigma) = Y_t^{-M'(\sigma_0)},$$

so that $Z_t(\sigma)$ is position at time t of the tagged particle started from $-M'$.

Given $V : \Omega \rightarrow \mathcal{M}(S_1)$ and $n \in \mathbb{N}$, let $V(\sigma, n)$ be the measure V shifted to the n 'th $+$ -particle right of the origin, i.e.

$$V(\sigma, n) := \tau_{-\tilde{n}} V(\sigma), \quad \text{with } \tilde{n} = \min \left\{ m : \sum_{i=1}^m \chi(\sigma(i) = +) = n \right\}$$

Then we have the identity

$$\tau_{-Y_t} V(\sigma_t) = \tau_{-Z_t(\sigma)} V(\sigma_t, N(\sigma_0)).$$

where $N(\sigma_0)$ is the number of $+$ -particles between $-M$ and 0 . Both M' and N are random and depend on σ_0 . Crucially however, they are independent of one another under the measure $\text{Ber}_p^0 := \text{Ber}_p(\cdot | \sigma_0(0) = +)$.

Let us consider the coupling measure $\mathbf{P}_{\mu[-M, \infty), 0}$, as defined at Eq. (11) except that σ_0^1 is conditioned to have $\sigma_0^1(0) = +$. Note that $Z_t(\sigma^1) = Z_t(\sigma^2)$ because $M'(\sigma_0^1) = M'(\sigma_0^2)$ and a tagged push particle started to the left of all discrepancies can never catch up with the discrepancies, see Proposition 4.2. Let A be the event that at time t the leftmost discrepancy is to the right of $Y_t(\sigma^1) + M$, where $Y_t(\sigma^1) = Y_t^0(\sigma^1)$ is the tagged particle started from the origin. On A , we have

$$\tau_{-Z_t(\sigma^1)} V(\sigma_t^1, N(\sigma_0^1)) = \tau_{-Z_t(\sigma^2)} V(\sigma_t^2, N(\sigma_0^1)) =: h(\sigma^2, \sigma_0^1) \quad (14)$$

since $Z_t(\sigma^1) = Z_t(\sigma^2)$ and $\sigma_t^1(x) = \sigma_t^2(x)$ for x smaller than the leftmost discrepancy.

Recall that by Proposition 4.2, the event A occurs with probability at least $1 - Ce^{c(M-t)}$. Using Eq. (14) and $1 = 1_A + 1_{A^c}$, we get

$$\begin{aligned} \mathbf{E}_{\text{Ber}_p, 0}[\tau_{-Y_t} V(\sigma_t) U(\sigma_0)] - \mathbf{E}_{\mu[-M, \infty), 0}[h(\sigma^2, \sigma_0^1) U(\sigma_0^1)] \\ = -\mathbf{E}_{\mu[-M, \infty), 0}[1_{A^c} (\tau_{-Y_t} V(\sigma_t) U(\sigma_0^1) - h(\sigma^2, \sigma_0^1) U(\sigma_0^1))] \end{aligned} \quad (15)$$

The second term on the left hand side may be re-expressed as

$$\int d\text{Ber}_p^0(\sigma_0^1) U(\sigma_0^1) \mathbf{E}_{\mu[-M, \infty), 0}[h(\sigma^2, \sigma_0^1) | \sigma_0^1(x), |x| \leq M]$$

The random variable $h(\sigma^2, \sigma_0^1)$ depends on σ_0^1 only through N , so we can conclude that

$$\begin{aligned} \mathbf{E}_{\text{Ber}_p, 0}[h(\sigma^2, \sigma_0^1)] &= \sum_{n \in \mathbf{N}} \chi\{N(\sigma_0) = n\} \mathbf{E}_{\text{Ber}_p, 0}[\tau_{-Y_t} V(\sigma_t, n)] \\ &= \sum_{n \in \mathbf{N}} \mathbf{E}_{\text{Ber}_p, 0}[\tau_{-Y_t} V(\sigma_t)] = 0. \end{aligned} \quad (16)$$

The second equality follows from translation invariance and the third follows since V is of zero mean. It follows that the second term on the left hand side in Eq. (15) vanishes and to conclude the proof, we need to estimate the total variation of the right hand side in Eq. (15), which is of the form $\tilde{\mathbf{E}}(J1_{A^c})$ with J a measure and $\tilde{\mathbf{E}} = \mathbf{E}_{\mu[-M, \infty), 0}$. We use

$$|\tilde{\mathbf{E}}(J1_{A^c})| \leq \tilde{\mathbf{E}}(|J|1_{A^c}) \leq \tilde{\mathbf{E}}(|J|^2)^{1/2} (\tilde{\mathbf{P}}(A^c))^{1/2},$$

with $|\cdot|$ denoting total variation and $\tilde{\mathbf{E}} = \mathbf{E}_{\mu[-M, \infty), 0}$. As already remarked, the probability of A^c is exponentially small, so we just need to bound $\tilde{\mathbf{E}}(|J|^2)^{1/2}$, which goes as follows:

$$\tilde{\mathbf{E}}(|J|^2(S_1 \times S_2))^{1/2} \leq 2\mathbf{E}_{\text{Ber}_p, 0}(|U|^4(S_1))^{1/4} \mathbf{E}_{\text{Ber}_p, 0}(|V|^4(S_2))^{1/4}$$

where we used stationarity of the process seen from the tagged particle. \square

5 Time-Reversal and the Adjoint Process

5.1 The Time-Reversal Map

Let us fix some time $\tau > 0$ and define the time-reversal map $\sigma \mapsto \tilde{\sigma}$ from $D_\Omega([0, \tau])$ to $D_\Omega([0, \tau])$ by

$$\tilde{\sigma}_s := \sigma_{(\tau-s)_-}, \quad 0 \leq s \leq \tau$$

This map is measurable and is one-to-one on a set of full $\mathbf{P}_{\text{Ber}_p}$ measure. Let $F \in \mathcal{B}(D_\Omega([0, \tau]))$ and consider the lift of the time-reversal map to functions $F \mapsto \tilde{F}$:

$$\tilde{F}(\sigma) := F(\tilde{\sigma}).$$

For each of our quasi-local processes X_t , we now have a time-reversed process \tilde{X}_t satisfying

$$\tilde{X}_t(\sigma) - \tilde{X}_s(\sigma) := -(X_{\tau-s}(\tilde{\sigma}) - X_{\tau-t}(\tilde{\sigma})), \quad 0 \leq s \leq t \leq \tau,$$

It is instructive to take $X_t = \int_0^t \chi\{\sigma_{s-}(x) = \eta\} dN_{x,\eta}(s)$. In this case, comparing $\sigma, \tilde{\sigma}$ at jump times, we get

$$d\tilde{X} = - \sum_{y>x} \chi_x^{-\eta} \chi_{(x,y]}^\eta(\sigma) dN_{y,\eta}(t).$$

This allows us to deduce that the mapping $X \rightarrow \tilde{X}$ maps quasi-local processes into quasi-local processes. The thing to keep in mind is that an arrival of $N_{x,\eta}$ at time s causing a jump for the process X corresponds to an arrival of $N_{y,\eta}$ at time $t - s$ for the process \tilde{X} where $y = \min(z : z > x, \sigma_{s-}(z + 1) = -\eta)$. More generally, with X determined by $(f, g_{x,\eta})$, the map $X \mapsto \tilde{X}$ corresponds to the map $(f, g_{x,\eta}) \rightarrow (\tilde{f}, \tilde{g}_{x,\eta})$ with

$$\tilde{f} = -f, \quad \tilde{g}_{y,\eta} = \sum_{x<y} g_{x,\eta} \chi_x^{-\eta} \chi_{(x,y]}^\eta.$$

The data $(\tilde{f}, \tilde{g}_{x,\eta})$ satisfy all necessary requirements:

Lemma 5.1. *If, as assumed throughout, $(f, g_{x,\eta})$ are such that all three Assumptions 2.1 2.2 and 2.3 are satisfied, then they are satisfied as well for $(\tilde{f}, \tilde{g}_{x,\eta})$.*

The straightforward verification of this lemma proceeds by using Holder inequalities and the fact that $\|\chi_{(x-r,x]}^\eta\|_{L^q(\text{Ber}_p)} \leq C e^{-cr}$ for any $q > 0$.

5.2 The Adjoint Process $\mathbf{P}_{\text{Ber}_p}^*$

Let us denote by $\mathbf{E}_{\text{Ber}_p}^*$ the expectation started from Ber_p of a left-moving Toom interface. Thus when the $N_{x,\eta}$ clock rings and $\sigma(x) = \eta$, we exchange the values of $\sigma(x), \sigma(y)$ with

$y := \max(z < x : \sigma(z) \neq \sigma(x))$. The left-moving process started from Ber_p is constructed analogously to the right moving process and again Ber_p is an invariant measure. The process can be started from σ Ber_p -almost surely, and we denote its expectation by \mathbf{E}_σ^* . The relation to the time-reversal map introduced above is that

$$\mathbf{E}_{\text{Ber}_p}[F] = \mathbf{E}_{\text{Ber}_p}^*[\tilde{F}] \quad (17)$$

Let us briefly sketch the verification of Eq. (17). First, using Remark 1.3 one verifies Eq. (17) for functions on $D([0, \epsilon])$ (note that on a finite cycle the corresponding statement is direct). Then using the Markov property and induction, one extends to functions on $D([0, \tau])$ for arbitrary τ

Here is the induction step: Assume Eq. (17) for functions of $D([0, t])$. We extend it to functions of $D([0, 2t])$. Let $s \in [0, 2t]$ and let f, g be bounded measurable functions. Then

$$\mathbf{E}[g(\sigma_0)f(\sigma_s)] = \mathbf{E}[g(\sigma_0)\mathbf{E}_{\sigma_{s/2}}[f(\sigma_{s/2})]] = \mathbf{E}^*[g(\sigma_{s/2})\mathbf{E}_{\sigma_0}[f(\sigma_{s/2})]].$$

The first equality follows from the Markov property while the second follows from the induction hypothesis for Eq. (17). Note here that the outer expectation corresponds to the left moving process while $\mathbf{E}_{\sigma_0}[f(\sigma_\epsilon)]$ corresponds to the right moving process. Using the Markov property again (for left moving process) the RHS is $\text{Ber}_p(\sigma_0)[E_{\sigma_0}^*g(\sigma_{s/2})E_{\sigma_0}[f(\sigma_{s/2})]]$. We are then done by symmetry ($E^*[g(\sigma_{2\epsilon})f(\sigma_0)]$ yields the same expression). The argument for a general finite product at different times in $[0, 2t]$ is similar. Then we conclude the induction step by density argument (or by the Monotone Class Theorem).

If we want to include the tagged particle, we begin by considering functions f_i on the extended state space $\Omega \times \mathbf{Z}$. It simplifies matters to assume that each f_i is translation covariant, i.e. $f_i(\sigma, y) = f_i(\tau_x\sigma, y - x)$, in which case Eq. (17) is upgraded to

$$\mathbf{E}_{\text{Ber}_p,0}[F] = \mathbf{E}_{\text{Ber}_p,0}^*[\tilde{F}] \quad (18)$$

Let us fix a time s and we consider two L^2 functions F_1, F_2 where $F_1 \in \mathcal{B}(\sigma_t, t \in [0, s])$ and $F_2 \in \mathcal{B}(\sigma_t, t \geq 0)$. Let

$$G(\sigma) := \mathbf{E}_\sigma^*(\tilde{F}_1).$$

Note that the $\tilde{\cdot}$ operation depends on a fiducial point τ , which is understood here to be $\tau = s$.

Lemma 5.2. *With s, F_1, F_2, G as above*

$$\mathbf{E}[F_1 F_2 \circ \theta_s] = \mathbf{E}[GF_2].$$

Proof. We have

$$\mathbf{E}_{\text{Ber}_p}[F_1 F_2 \circ \theta_s] = \mathbf{E}_{\text{Ber}_p}[F_1 \mathbf{E}_{\sigma_s}[F_2]] \quad (19)$$

$$= \int d\text{Ber}_p(\sigma) \mathbf{E}_{\text{Ber}_p}[F_1 | \sigma_s = \sigma] \mathbf{E}_\sigma[F_2] \quad (20)$$

$$= \mathbf{E}_{\text{Ber}_p}[GF_2] \quad (21)$$

Here the first equality is due to the Markov property, the second is due to stationarity of Ber_p and the definition of the conditional expectation. The third equality follows from the fact that $\mathbf{E}_\sigma^*(\tilde{F}_1)$ is a version of $\mathbf{E}_{\text{Ber}_p}(F_1 | \sigma_s = \sigma)$ cf. Eq. (17). \square

5.3 An Application

To foreshadow future applications, we use Lemma 5.2 to obtain identities between measures generated by ℓ quasi-local processes $X^{(j)}(t)$. Let us first assume that $(X^{(j)})_{j=1,\dots,\ell}$ are quasi-local w.r.t. to the origin. Let $l \in [1, \ell]$ and observe that

$$\mathbf{E}_{\text{Ber}_p}[dX^{(1)}(t_1) \dots dX^{(\ell)}(t_\ell)] = dt_l \mathbf{E}_{\text{Ber}_p}[H(dt_{1,\dots,l-1}) dX^{(l+1)}(t_{l+1}) \dots dX^{(\ell)}(t_\ell)]$$

where $H(dt_{1,\dots,l-1}) = H(\sigma, dt_{1,\dots,l-1})$ is the random variable on Ω , defined Ber_p -a.s., with values in measures on $\mathcal{M}(\mathcal{S}_{l-1})$, cf. Lemma 2.7 given by

$$H(\sigma, dt_{1,\dots,l-1}) = \mathbf{E}_{\text{Ber}_p}[dX^{(1)}(t_1) \dots dX^{(l-1)}(t_{l-1}) | \sigma_{t_l} = \sigma]$$

It is useful to rewrite this formula using the adjoint process. Consider the change of variables

$$s = t_l, \quad w_j = t_l - t_{l-j}, \quad \text{for } j = 1, \dots, l-1 \quad u'_j = t_{j+l} - t_l, \quad \text{for } j = l+1, \dots, \ell.$$

We will work in Sections 6.2 and 7 with this change of variables.

Then using stationarity (and abusing the notation for H), on the set $\mathcal{A} := \{0 < w_1 < \dots < w_{l-1} < s, 0 < u'_1 < \dots < u'_{\ell-l}\}$

$$\mathbf{E}_{\text{Ber}_p}[dX^{(1)}(t_1) \dots dX^{(\ell)}(t_\ell)] \mapsto ds \mathbf{E}_{\text{Ber}_p}[H(\sigma, dw_{1,\dots,l-1}) dX^{(l+1)}(u'_1) \dots dX^{(\ell)}(u'_{\ell-l})]. \quad (22)$$

Here the measure $H(\sigma, \cdot)$ satisfies

$$\begin{aligned} H(\sigma, dw_{1,\dots,l-1}) &= \tilde{f}(\sigma) \mathbf{E}_\sigma^*(d\tilde{X}^{(l-1)}(w_1) \dots d\tilde{X}^{(1)}(w_{l-1})) \\ &\quad + \sum_{x,\eta,r} \chi_r^{-\eta} \chi_{(x-r,x]}^\eta \tilde{g}_{x,\eta}^{(l)}(\sigma) \mathbf{E}_{\sigma^{x-r,x}}^*(d\tilde{X}^{(l-1)}(w_1) \dots d\tilde{X}^{(1)}(w_{l-1})) \end{aligned} \quad (23)$$

where r ranges over $r = 0, 1, 2, \dots$ and $\sigma^{x-r,x}$ is obtained from σ by exchanging $\sigma(x-r), \sigma(x)$.

Eventually, we are interested in the total variation over the set $E_l(T, \kappa)$ (see Eq. (7)), which under the change of variables gives the restrictions $w_{l-1} < T$ and $u'_1 > T^\kappa$ on \mathcal{A} (among other conditions). It is convenient on $\{u'_1 > T^\kappa\}$ to change variables once more. Applying the Markov property and setting $u_i = u'_i - T^\kappa$, our measure transforms into

$$\mathbf{E}_{\text{Ber}_p}[\mathrm{d}X^{(1)}(t_1) \dots \mathrm{d}X^{(\ell)}(t_\ell)] \mapsto \mathrm{d}s \mathbf{E}_{\text{Ber}_p}[H(\sigma, \mathrm{d}w_{1,\dots,l-1})K(\sigma_{T^\kappa}, \mathrm{d}u_{1,\dots,\ell-l})] \quad (24)$$

where the measure K is given by

$$K(\sigma, \mathrm{d}u_{1,\dots,\ell-l}) = \mathbf{E}_\sigma[\mathrm{d}X^{(l+1)}(u_1) \dots \mathrm{d}X^{(\ell)}(u_{\ell-l})]$$

and

$$E_l(T, \kappa) \mapsto \{0 < w_1 < \dots < w_{l-1} < T < s, 0 < u_1 < \dots < u_{\ell-l} < L - T^\kappa - s\} =: E'. \quad (25)$$

If we consider $X^{(j)}$ quasi-local w.r.t. to the tagged particle, then we can write the same formulas as above provided we replace $\mathbf{E}_{\text{Ber}_p}, \mathbf{E}_\sigma^*, \mathbf{E}_{\sigma^{x-r,x}}^*$ with $\mathbf{E}_{\text{Ber}_p,0}, \mathbf{E}_{\sigma,0}^*, \mathbf{E}_{\sigma^{x-r,x},0}^*$ and K by its natural analog depending on σ_{T^κ} and Y_{T^κ} .

6 Bounds on Localization of Conditional Expectations

6.1 A General Principle

Let the function F satisfy $F \in \mathcal{B}(\sigma_s(x), (x, s) \in A)$ for some Borel set $A \subset \mathbf{Z} \times \mathbf{R}_+$. Let

$$G(\sigma) := \mathbf{E}_\sigma(F)$$

and define its local approximations by

$$\mathbf{P}_R G(\sigma) := \text{Ber}_p \left[G(\cdot) | \sigma_0(j) = \sigma(j) : j \in [-R, R] \right].$$

We define the event

$$E_A := \{x \in \mathbf{D}_t \text{ for some } (x, t) \in A\}.$$

i.e. the event that there is a discrepancy in A . Let ν_k be defined as the measure on pairs of spin configurations $\sigma = (\sigma^1, \sigma^2)$ such that:

- i) The marginal distributions of σ^1 and σ^2 are Ber_p .
- ii) For $j \neq \pm(k+1)$, $\sigma^1(j) = \sigma^2(j)$ ν_k -a.s.
- iii) For $j = \pm(k+1)$, $\sigma^1(j), \sigma^2(j)$ are independent.

We can now state the main result of this section.

Lemma 6.1. *For any $R > 0$*

$$\text{Ber}_p \left[(G - P_R G)^2 \right] \leq C \sqrt{\mathbf{E}_{\text{Ber}_p}[F^4]} \sum_{k \geq R} \sqrt{\mathbf{P}_{\nu_k}[E_A]}$$

For the tagged particle case, we get the same lemma with the replacements

$$\begin{aligned} F \in \mathcal{B}(\sigma_s(x) : (x, s) \in A) &\rightarrow F \in \mathcal{B}(\sigma_s(x), Y_s : (x, s) \in A), \\ G(\sigma) = \mathbf{E}_\sigma(F) &\rightarrow G(\sigma) = \mathbf{E}_{\sigma,0}(F), \\ \mathbf{P}_{\text{Ber}_p} &\rightarrow \mathbf{P}_{\text{Ber}_p,0}. \end{aligned}$$

Proof. Using the natural decomposition of in terms of martingale increments,

$$\text{Ber}_p \left[(G - P_R G)^2 \right] = \sum_{k \geq R} \text{Ber}_p \left[(P_k G - P_{k+1} G)^2 \right]$$

We represent

$$P_k G(\sigma) - P_{k+1} G(\sigma) = \nu_{\sigma,k} [G(\sigma^1) - G(\sigma^2)] = \mathbf{E}_{\nu_{\sigma,k}} [F(\sigma^1) - F(\sigma^2)] \quad (26)$$

where, for $\sigma \in \Omega$, $\nu_{\sigma,k}$ is the probability measure on Ω^2 (with configurations (σ^1, σ^2)) such that:

- i) $\sigma^1(x) = \sigma^2(x) = \sigma(x)$ for $|x| \leq k$ and for $x = \pm(k+1)$, $\sigma^2(x) = \sigma(x)$.
- ii) The variables $\sigma^1(x), \sigma^2(x)$ for $|x| > k+1$ and $\sigma^1(x)$ for $x = \pm(k+1)$ are i.i.d. They are $+1$ with prob p and -1 with prob $1-p$.

By Eq. (26) and the Cauchy-Schwartz inequality,

$$\text{Ber}_p [(P_k G - P_{k+1} G)^2] \leq \mathbf{E}_{\nu_k} [(F(\sigma^1) - F(\sigma^2))^2].$$

The utility of this last inequality is to reduce the derivation of error bounds in the localization of G to controlling the behavior of a pair of discrepancies. By assumption on F , for any fixed k ,

$$\mathbf{E}_{\nu_k} [(F(\sigma^1) - F(\sigma^2))^2] = \mathbf{E}_{\nu_k} [(F(\sigma^1) - F(\sigma^2))^2 1_{E_A}] \quad (27)$$

so that

$$\text{Ber}_p [(P_k G - P_{k+1} G)^2] \leq C \sqrt{\mathbf{E}_{\text{Ber}_p}[F^4]} \sqrt{\mathbf{P}_{\nu_k}(E_A)} \quad (28)$$

since the marginals of ν_k are Ber_p . □

6.2 An Application to the Random Measure H

Continuing the discussion from Section 5.3, we want to apply Lemma 6.1 to argue that the random measure $H(\sigma, dt_{1,\dots,l})$ can be localized in σ . With a view toward the justification of Lemma 2.8, we shall bound the variation of H on

$$\mathcal{S}_{l-1}(T)$$

To state the main point of this section, let us first extend the action of the projection/conditional expectation P_R to random measures as follows. For ν a random measure taking values in $\mathcal{M}(\mathcal{S}_\ell(L))$, we define $P_R\nu$ by

$$\int h P_R(\nu) := P_R \left(\int h \nu \right)$$

for all bounded continuous functions h on $\mathcal{S}_\ell(L)$. To check that this is a meaningful definition, note that

$$\left| P_R \left(\int h \nu \right) \right| \leq P_R(|\nu|(\mathcal{S}_\ell(L))) \|h\|_\infty$$

which is finite almost surely, since $|\nu|(\mathcal{S}_\ell(L))$ is finite almost surely. That the random measure is well-defined then follows from Lemma 2.7.

Lemma 6.2. *Let $\kappa > 1$. For $R \geq T^\kappa$,*

$$\|H - P_R H|(\mathcal{S}_{l-1}(T))\|_{L^q} \leq C e^{-cT^{\frac{\kappa-1}{2}}}$$

Starting from the expression for H , Eq. (23), the main technical issue in proving Lemma 6.2 is to deal with the non-locality of the measure

$$Z(dw_{1\dots l-1}) = Z(\sigma, dw_{1\dots l-1}) := \mathbf{E}_\sigma^*[d\tilde{X}^{(l-1)}(w_1) \dots d\tilde{X}^{(1)}(w_{l-1})].$$

As such, we first study this expression separately. The main application of Lemma 6.1 is the following.

Lemma 6.3. *With T, R, α as in Lemma 6.2,*

$$\|Z - P_R Z|(\mathcal{S}_{l-1}(T))\|_{L^q} \leq C e^{-cT^{\frac{\kappa-1}{2}}}.$$

Proof of Lemma 6.3. Let us denote by $Z_{R/2}$ the result of replacing all \tilde{X} by their localized versions $\tilde{X}_{R/2}$, see Section 3. Abbreviating $\mathcal{S}_{l-1}(T) = \mathcal{S}$ and using the triangle inequality,

$$\|Z - P_R Z|(\mathcal{S})\|_{L^q} \leq \|Z_{R/2} - P_R Z_{R/2}|(\mathcal{S})\|_{L^q} + \|Z - Z_{R/2}|(\mathcal{S})\|_{L^q} + \|P_R Z_{R/2} - P_R Z|(\mathcal{S})\|_{L^q}.$$

We observe that the last contribution on the RHS is bounded by the second contribu-

tion by applying Jensen's inequality for conditional expectations. To bound the second contribution, we have, writing $\nu^{(j)}$ for the measure $d\tilde{X}^{(j)}$,

$$\|Z_{R/2} - Z|(\mathcal{S})\|_{L^q} \leq C \left\| \mathbf{E}_\sigma^* \left[\sum_i |\nu^{(i)} - \nu_{R/2}^{(i)}|[0, T] \prod_{j \neq i} |\nu^{(j)}|[0, T] \right] \right\|_{L^q}.$$

Using Jensen's inequality and then Hölder's inequality,

$$\|Z_{R/2} - Z|(\mathcal{S})\|_{L^q} \leq C \max_{j \leq l-1} \|\nu^{(j)} - \nu_{R/2}^{(j)}|[0, T]\|_{L^{q(l-1)}} \prod_{j=1}^{l-1} (1 + \|\nu^{(j)}|[0, T]\|_{L^{q(l-1)}}).$$

By Lemma 3.1, the RHS is bounded by Ce^{-cR} (which is sufficient for the claimed bound of Lemma 6.3). Therefore we reduce the proof to providing bounds on $Z_{R/2} - \mathbf{P}_R Z_{R/2}$.

To put ourselves in the framework of Lemma 6.1 observe that the conclusion of said lemma applies equally to the adjoint process by symmetry and recall that the variation of a measure on $\mathcal{S}_{l-1}(T)$ may be characterized by

$$|Z_{R/2} - \mathbf{P}_R Z_{R/2}|(\mathcal{S}_{l-1}(T)) = \sup_{h: \|h\|_\infty=1} |G_h - \mathbf{P}_R G_h|.$$

where $G_h := \int_{\mathcal{S}_{l-1}(T)} h Z_{R/2}$ for a bounded continuous h . The role of F in Lemma 6.1 is played here by $F = F_h = \int h d\tilde{X}_{R/2}^{(l-1)}(w_1) \dots d\tilde{X}_{R/2}^{(1)}(w_{l-1})$. By the a priori bound Lemma 3.1 we estimate $(\mathbf{E}[F^4])^{1/4}$ by $C\|h\|_\infty T^{l-1}$. The role of the set A is played by

$$A = \{(s, x) : s \in [0, T], |x| \leq R\} \cup \{(s, x) : s \in [0, T], |x - Y_s| \leq R\}$$

Note that A is random here, so we actually need a straightforward generalization of Lemma 6.1 which is omitted. The probability $\mathbf{P}_{\nu_k}^*(E_A)$ is the probability that at least one of the discrepancies started at $\pm(k+1)$ comes closer than $R/2$ to the tagged particle in $[0, T]$ or that a discrepancy enters the region $[-R/2, R/2]$ in time $[0, T]$. For the discrepancy started at $k+1$, we simply use Proposition 4.2 to argue that the tagged particle can not catch up. For the discrepancy started at $-k-1$, we use the maximal speed of discrepancy and tagged particle, see Lemmas 4.4 and 4.5. In particular, if $k > R$, and recalling $R > T^\kappa, \kappa > 1$, we get

$$\mathbf{P}_{\nu_k}^*[E_A] \leq Ce^{-c(k/T)^{1/2}}.$$

Performing the sum $\sum_{k>R} (\mathbf{P}_{\nu_k}^*[E_A])^{1/2}$ we get $Ce^{-cT^{\frac{\kappa-1}{2}}}$. This yields the required bound on the variation. \square

Proof of Lemma 6.2. It remains to pass from estimates on $Z - \mathbf{P}_R Z$ to estimates on

$H - H_R$. This is done by telescoping:

$$\begin{aligned} |(H - \mathbf{P}_R H)(\sigma)| &\leq |(f^{(l)} - \mathbf{P}_R f^{(l)})(\sigma)| |Z(\sigma)| + |\mathbf{P}_R f^{(l)}(\sigma)| |(Z - \mathbf{P}_R Z)(\sigma)| \\ &+ \sum_{\beta} |(v_{\beta} - \mathbf{P}_R v_{\beta})(\sigma)| |Z(\sigma^{\beta})| + \sum_{\beta} |\mathbf{P}_R v_{\beta}(\sigma)| |(Z - \mathbf{P}_R Z)(\sigma^{\beta})|. \end{aligned} \quad (29)$$

where the index $\beta = (x, \eta, r) \in \mathbf{Z} \times \{1, -1\} \times \mathbf{N}$ and we have abbreviated

$$v_{\beta} = \chi_r^{-\eta} \chi_{(x-r, x]}^{\eta} \tilde{g}_{x, \eta}^{(l)}, \quad \sigma^{\beta} = \sigma^{x-r, x}.$$

We bound only the third term explicitly (the rest are simpler or similar to handle), call it V .

$$V(\sigma) \leq \sum_{\beta} |(v_{\beta} - \mathbf{P}_R v_{\beta})(\sigma)| |Z(\sigma^{\beta})| (\mathcal{S}_{l-1}(T)). \quad (30)$$

By the triangle inequality, then Cauchy Schwarz

$$\|V(\mathcal{S}_{l-1}(T))\|_{L^q} \leq \sum_{\beta} \|v_{\beta} - \mathbf{P}_R v_{\beta}(\sigma)\| |Z(\sigma^{\beta})| (\mathcal{S}_{l-1}(T)) \|_{L^q} \quad (31)$$

$$\leq \sup_{\beta} \|Z(\sigma^{\beta})| (\mathcal{S}_{l-1}(T))\|_{2q} \sum_{\beta} \|v_{\beta} - \mathbf{P}_R v_{\beta}\|_{L^{2q}} \quad (32)$$

The L^{2q} -norm of $Z(\sigma^{\beta})| (\mathcal{S}_{l-1}(T))$ is bounded independently of β by CT^{l-1} by Lemma 3.1, and remaining sum over β is bounded by e^{-cR} using Assumptions 2.1 and 2.2. This yields the claim. \square

6.3 An Application to the random measure K

We localize the measure K too, though in a slightly different sense than for H . Let

$$P_{(-\infty, R]} K(\sigma) = \text{Ber}_p(K \mid \sigma_x, x \leq R)$$

Lemma 6.4.

$$|P_{(-\infty, R]} K - K| \leq C e^{-cR} L^{\ell-l} \quad (33)$$

Proof. First we note that we can change K into $K_{R/2}$ (i.e. replacing $X^{(i)}$ by $X_{R/2}^{(i)}$ at the expense of an error of order $Ce^{-cR}L^{\ell-l}$ in total variation. Just as for H , this is an application of Lemma 3.1, see Lemma 6.3. It remains then to estimate $|P_{(-\infty, R]} J - J|$ with $J = K_{R/2}$. We remark that, if $J(\sigma) = \mathbf{E}_{\sigma}(F)$ with $F \in \mathcal{B}(\sigma_s(x), x \in \mathbf{Z}, s \geq 0)$, then

$$\text{Ber}_p^0[|P_{(-\infty, R]} J - J|^q] \leq \mathbf{E}_{\mu(R, \infty), 0}[|F(\sigma^1) - F(\sigma^2)|^q] \quad (34)$$

If moreover $F \in \mathcal{B}(\sigma_s(x), x - Y(s) \leq R/2)$, then

$$\mathbf{E}_{\mu(R, \infty), 0}[|F(\sigma^1) - F(\sigma^2)|^q]^{1/q} \leq C \|F\|_{L^{2q}} \mathbf{P}(E)^{1/(2q)}$$

with E the event that under $\mathbf{E}_{\mu(R, \infty), 0}$, the leftmost discrepancy remains a distance $R/2$ to the right of $Y(t)$ for all times. $\mathbf{P}(E) \leq Ce^{-cR}$ by Proposition 4.2. \square

6.4 Proof of Lemma 2.8

We recall that we are out to bound the variation of the measure

$$ds \mathbf{E}_{\text{Ber}_p}[H(\sigma; dw_{1, \dots, l-1})K(\sigma_{T^\kappa}, du_{1, \dots, u_{\ell-l}})].$$

on the set E' defined at Eq. (25).

Let us fix $s > 0$ and estimate the restricted measure uniformly in this variable. In this case (and when restricted to the relevant subspace of E'), H is a measure on $S_1 := \mathcal{S}_{l-1}(T)$ and K is a measure on $S_2 := \mathcal{S}_{\ell-l}[T^\kappa, L]$. We replace H, K by $P_R H, P_{(\infty, R]} K$. These substitutions make an error in the total variation of order $CL^{\ell-l+1}(e^{-cR} + e^{-cT^\alpha})$, see Lemma 6.2 and Lemma 6.4. Then we are down to estimating

$$\mathbf{E}(P_R H(\sigma_0) P_{(\infty, R]} K(\sigma_{T^\kappa})) - \mathbf{E}(P_R H(\sigma_0)) \mathbf{E}(P_{(\infty, R]} K(\sigma_{T^\kappa}))$$

and this is now in the form of Lemma 4.7. This ends the proof of Lemma 2.8.

7 Finite Dimensional Convergence and Tightness

Having established Lemma 2.8, we are ready to derive the various functional CLTs. By adding a constant drift, it suffices to consider only quasi-local processes X with $\mathbf{E}(X(t)) = 0$. For such X , we consider $X_n(\cdot) := 1/\sqrt{n}X(\cdot n)$ and we prove first that for a fixed sequence of times $t_1 < \dots < t_l$ the vector

$$1/\sqrt{n}(X_n(t_1), \dots, X_n(t_l)) \tag{35}$$

converges weakly to the appropriate multivariate Gaussian, see Theorem 7.1. Then we argue that the sequence of processes $(X_n)_{n \in \mathbf{N}}$ is tight in the Skorohod space $D([0, 1], \mathbf{R})$, see Proposition 7.2. By standard reasoning, these two results complete the proof of our main result 2.5. The rest of this section is hence devoted to the proof of these results.

We first compute the $t \rightarrow \infty$ limit of the variance of $(1/\sqrt{t})X(t)$. It is given as

$D = D_X := D_1 + D_2$ with

$$D_1 := \sum_{x,\eta} \lambda_\eta \mathbf{E}((g_{x,\eta}(0))^2) = \sum_{x,\eta} \lambda_\eta \text{Ber}_p(g_{x,\eta}^2),$$

$$D_2 := 2 \int_0^\infty \mathbf{E}[h(s)h(0)]ds, \quad h = f + \sum_{x,\eta} \lambda_\eta g_{\eta,x}$$

It is a straightforward consequence of Proposition 4.1 or Proposition 4.2 that $D < \infty$.

To prove convergence of finite-dimensional distributions, we use the method of moments. Let us consider increasing sequences $(a_i)_{i \leq k}, (b_i)_{i \leq k} \in \mathbf{R}^k$ such that $0 \leq a_i < b_i < a_{i+1}$ and let

$$\Delta_i X_n = X_n(b_i) - X_n(a_i) = \frac{1}{\sqrt{n}}[X(b_i n) - X(a_i n)],$$

Let $\gamma := (\gamma_i)_{i=1}^k \in \mathbf{R}^k$ and let $(N_i)_{i=1}^k$ be independent mean zero Gaussians with respective variance $D[b_i - a_i]$.

Theorem 7.1. *For all $\ell \in \mathbf{N}$ and $\epsilon > 0$, there is $C(\epsilon, \ell) > 0$ such that*

$$\left| \mathbf{E}[(\gamma \cdot \Delta X_n)^\ell] - \mathbf{E}[(\gamma \cdot N)^\ell] \right| \leq C n^{\epsilon-1/2}.$$

This implies (method of moments) that the vector (35) converges in distribution to

$$\sqrt{D}(B(t_1), \dots, B(t_l)),$$

where $B(t)$ is a standard Brownian motion.

The tightness is also in essence a consequence of the above theorem.

Proposition 7.2. *The sequence $(X_n)_{n \in \mathbf{N}}$ is tight in $D_\Omega([0, 1])$, equipped with the Skorohod topology.*

Proof. We first fix some notation. For any $1 > \delta > 0$, we fix a partition $\mathcal{J}(\delta)$ of $[0, 1]$ by intervals I_j with lengths between δ and 2δ . For any interval I we write

$$w_X(I) = \sup_{s,t \in I} |X(t) - X(s)|$$

Tightness of the sequence X_n is implied by the following two conditions (see e.g. [1])

1. For any $\kappa, \epsilon > 0$, there is a $1 > \delta > 0$ such that

$$\limsup_n \mathbf{P}(\max_{I \in \mathcal{J}(\delta)} w_{X_n}(I) \geq \epsilon) \leq \kappa$$

2. For any $\eta > 0$, there is an M such that

$$\sup_n \mathbf{P} \left(\sup_{0 \leq t \leq 1} |X_n(t)| \geq M \right) \leq \eta$$

Now, we check these conditions, starting with 1).

As in Section 3, we denote by \bar{X} the quasi-local process obtained derived from X by replacing (f, g) by $(|f|, |g|)$. Then clearly $\bar{X}(t)$ is increasing and so

$$\sup_{0 \leq s \leq t \leq u} |X(t) - X(s)| \leq \bar{X}(u) \quad \forall u \in \mathbf{R}_+.$$

Since all X_n are stationary quasi-local processes, we then find

$$\mathbf{P} \left(\max_{I \in \mathcal{J}(\delta)} w_{X_n}(I) \geq \epsilon \right) \leq C \delta^{-1} \mathbf{P}(\bar{X}_n(2\delta) \geq \epsilon) \quad (36)$$

To bound the probability on the right hand side, we use Theorem 7.1 for \bar{X} and $\ell = 4$;

$$\mathbf{E}(\bar{X}_n^4(t)) \leq C(t^2 + n^{-1/4})$$

so that, by (36) and the Markov inequality, we get

$$\mathbf{P} \left(\max_{I \in \mathcal{J}(\delta)} w_{X_n}(I) \geq \epsilon \right) \leq C \delta^{-1} \frac{(\delta^2 + n^{-1/4})}{\epsilon^4}$$

which settles condition 1) above. To check condition 2), it suffices to again consider the increasing $\bar{X}_n(t)$ and to establish $\mathbf{E}((\bar{X}_n(1))^2) < C$. The latter follows again by Theorem 7.1.

□

Proof of Theorem 7.1. Much of the work done here is (standard) combinatorics to suitably reduce (by expanding) the moments to expressions we can more easily compute. Let us fix the time scale n . For simplicity, we first do the case $k = 1$. We set $L = b_1 - a_1$ and by stationarity we can restrict to the interval $[0, L]$. We start from

$$\mathbf{E} \left[(\gamma \Delta X_n)^\ell \right] = n^{-\ell/2} \gamma^\ell \int_{[0, L]^\ell} \mathbf{E} \left[\prod_{j=1}^{\ell} dX(t_j) \right] \quad (37)$$

The measure $\mathbf{E} \left[\prod_{j=1}^{\ell} dX(t_j) \right]$ is not absolutely continuous due to singular contributions on diagonals $t_j = t_{j'}$. Formally, this comes about because the powers $(dX(t))^q$ are not necessarily zero. We find it computationally convenient to further reduce the problem to a sum of stochastic integrals over the open simplexes $\mathcal{S}_r(nL)$, with $r \leq \ell$ by viewing the

$(dX(t))^q$ as quasi-local processes themselves.

To make this part of the expansion explicit, we introduce more notation: Let $\mathbf{j} = \{j(1), \dots, j(r)\}$ denote a partition of ℓ , i.e. $j(l) \in \mathbf{N}$ and $\sum_{l=1}^r j(l) = \ell$. Then

$$\int_{[0,L]^\ell} \mathbf{E} \left[\prod_{j=1}^\ell dX(t_j) \right] = \sum_{\mathbf{j}} \frac{\ell!}{j(1)! \dots j(r)!} \mathbf{E} [Z(\mathbf{j})] \quad (38)$$

where

$$Z(\mathbf{j}) = \int_{\mathcal{S}_r(nL)} dW^{(1)}(t_1) \dots dW^{(r)}(t_r), \quad \text{with} \quad dW^{(l)}(t) := (dX(t))^{j(l)} \quad (39)$$

The $W^{(l)}$'s may have increments with nonzero mean. To give a clean statement below, let $d\mathcal{W}^{(l)}(t) = dW^{(l)}(t) - \mathbf{E}[dW^{(l)}(t)]$. Expanding Eq. (39) gives

$$\mathbf{E}[dW^{(1)}(t_1) \dots dW^{(r)}(t_r)] = \sum_{A \subset [r]} \prod_{l' \in A^c} \mathbf{E} [dW^{(l')}(t_{l'})] \mathbf{E} \left[\prod_{l \in A} d\mathcal{W}^{(l)}(t_l) \right] \quad (40)$$

Now we will use input from the previous sections, in particular Lemma 2.8, to calculate the leading contribution to $\mathbf{E} [\prod_{l \in A} d\mathcal{W}^{(l)}(t_l)]$.

Lemma 7.3 (Iterative Decomposition of Correlations). *Fix $L \in (0, 1)$, $n \in \mathbf{N}$. Let $(\mathcal{W}^{(l)})_{l=1}^r$ be quasi-local observables having mean zero increments and respective integrands $(f^{(l)}, g^{(l)})_{l=1}^r$. Let r be odd, then*

$$\left| \mathbf{E} \left[\prod_{l=1}^r d\mathcal{W}^{(l)}(t_l) \right] \right|_{\mathcal{S}_r(bn)} \leq C n^{r/2-1/2+\epsilon}, \quad \text{for any } \epsilon > 0, \quad (41)$$

where $|\cdot|_{\mathcal{S}_r(nL)}$ is the total variation on the simplex $\mathcal{S}_r(nL)$. For even r , we have

$$\left| \mathbf{E} \left[\prod_{l=1}^r d\mathcal{W}^{(l)}(t_l) \right] - \prod_{l < r, l \text{ odd}} \mathbf{E} [d\mathcal{W}^{(l)}(t_l) d\mathcal{W}^{(l)}(t_{l+1})] \right|_{\mathcal{S}_r(bn)} \leq C n^{r/2-1+\epsilon}. \quad (42)$$

Furthermore, the total variation of $\mathbf{E} [d\mathcal{W}^{(l)}(0) d\mathcal{W}^{(l+1)}(t)]$ on $\{t \geq T\}$ is bounded by $C e^{-cT^{1/4}}$.

This lemma will be proved after the proof of Theorem 7.1 is concluded.

We are now ready to determine, from among the terms expanded in Eqs. (37), (38), (39) and (40), the main contributions to the ℓ 'th moment. We keep \mathbf{m} fixed and we compute the contribution from the relevant \mathbf{j} 's and A 's. From Lemma 7.3 we deduce that

the maximal contribution to Eq. (40) is of order

$$n^{|A^c|} n^{|A|/2},$$

provided that *i*) $|A|$ is even, *ii*) for any odd $l \in A$, there is no $l' \in A^c$ such that $t_l \leq t_{l'} \leq t_{l+1}$, *iii*) for all $l' \in A$, the increment $dW^{(l')}$ has nonzero mean. Subleading contributions are down by at least a factor $n^{-1/2+\epsilon}$. Looking back at Eq. (39) and recalling that dX had zero mean, we see that the leading contributions are of order $n^{\ell/2}$, for ℓ even, and they occur when all $j(l)$ are either 1 or 2, and for each time t_l for which $j(l) = 1$, there is a partner time $t_{l'}$ such that $j(l') = 1$ and $|l - l'| = 1$. The pairs (l, l') are those that constitute the sets A for the dominant contributions in Lemma 7.3. Let $q = |\{l : j(l) = 2\}|$. Then the above considerations lead to

$$\mathbf{E}[Z(\mathbf{j})] = \frac{(nL)^{\ell/2}}{(\ell/2)!} D_1^q (D_2/2)^{\ell/2-q} + O(n^{\ell/2-1/2+\epsilon}).$$

where we also used that

$$\int_{\mathcal{S}_2(T)} E[dX(t_1)dX(t_2)] = \frac{1}{2} D_2 T + \mathcal{O}(Ce^{-cT^{1/4}}), \quad \int_0^T E[(dX(t))^2] = D_1 T$$

After summing over leading \mathbf{j} in (38), we arrive at (for ℓ even, otherwise we get only the error term)

$$\mathbf{E}[(\gamma \Delta X_n)^\ell] = (\gamma^2 LD)^{\ell/2} \frac{l!}{(l/2)! 2^{\ell/2}} + O(n^{\ell/2-1/2+\epsilon})$$

Recognizing the ℓ 'th moment of a Gaussian on the right hand side concludes the proof for the case $k = 1$. For general k , we proceed similarly, but with obvious restrictions on the range of time-arguments in the $dX(t_j)$. The only change that deserves a comment is the case where in (42), there appear pairs t_l, t_{l+1} such that one of them belongs to $[a_i, b_i]$ and the other to $[a_{i'}, b_{i'}]$ with $i \neq i'$. Let us pretend that $a_{i'} = b_i$ (other possibilities are easier to handle). Contributions of such pairs are subleading by the decay of correlation function (last claim of Lemma 7.3). □

Proof of Lemma 7.3. Define first the sequence of numbers $v_i, i = 1, 2, \dots$ recursively by

$$v_1 = \log^4 n, \quad v_{i+1} = \left(\sum_{j=1}^i v_j \right)^2$$

The main idea is to decompose the simplex $\mathcal{S}_r(nL)$ in clusters by grouping consecutive times. We fix an increasing sequence $(t_1, \dots, t_r) \in \mathcal{S}_r(nL)$ and we define a grouping of the times t_i in clusters (in fact, this is simply a grouping of the indices $1, \dots, r$). We let

$\mathcal{T}_1 := \{t_1, t_2, \dots, t_{z_1}\}$ where $z_1 > 1$ is the first index for which

$$(t_{z_1+1} - t_{z_1}) > v_{z_1}, \quad \text{or } z_1 = r$$

Once \mathcal{T}_1 defined (and $z_1 \neq r$), we define \mathcal{T}_2 by deleting the times $t_{\mathcal{T}_1}$ from the sequence (t_1, \dots, t_r) , renumbering the remaining ones, and repeating the above step. More concretely, we set $\mathcal{T}_2 := \{t_{z_1+1}, \dots, t_{z_2}\}$ where $z_2 > z_1$ is the first index for which

$$(t_{z_2+1} - t_{z_2}) > v_{z_2-z_1}, \quad \text{or } z_2 = r$$

This is repeated until we get a cluster decomposition

$$\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_d)$$

of (t_1, t_2, \dots, t_r) (some clusters can be singletons). The (sole) important properties of this cluster decomposition are

- a) All times in a given cluster are close to each other: $\max \mathcal{T}_i - \min \mathcal{T}_i \leq C \log^C n$, where $C = C(r)$
- b) The distance from \mathcal{T}_{i+1} to \mathcal{T}_i is large compared to the length of \mathcal{T}_i : There is some $T \geq \log^4 n$, depending only on the number of times in \mathcal{T}_i , such that

$$\max \mathcal{T}_i - \min \mathcal{T}_i < T, \quad (\min \mathcal{T}_{i+1} - \max \mathcal{T}_i) \geq T^2$$

A cluster decomposition $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_d)$ defines naturally a subset of $\mathcal{S}_r(nL)$ that we call $\mathcal{S}_{\mathcal{T}}$. We now draw two conclusions from previous estimates, that follow from these respective properties

1. The total variation of the measure $\mathbf{E} \left[\prod_{l=1}^r d\mathcal{W}^{(l)}(t_l) \right]$ on $\mathcal{S}_{\mathcal{T}}$ is bounded by $Cn^d \log^C n$ with d the number of clusters in the cluster decomposition \mathcal{T} . This is a consequence of the a-priori estimate of Lemma 3.1.
2. The measure factorizes on clusters, up to a small error:

$$\left| \mathbf{E} \left[\prod_{l=1}^r d\mathcal{W}^{(l)}(t_l) \right] - \prod_j \mathbf{E} \left[\prod_{t_l \in \mathcal{T}_j} d\mathcal{W}^{(l)}(t_l) \right] \right|_{\mathcal{S}_{\mathcal{T}}} \leq Cn^r e^{-\log^2 n}$$

This follows inductively from the crucial Lemma 2.8 by using the property b) above. Indeed, by direct application of Lemma 2.8 with $\kappa = 2$, we get that

$$\left| \mathbf{E} \left[\prod_{l=1}^r d\mathcal{W}^{(l)}(t_l) \right] - \mathbf{E} \left[\prod_{t_l \in \mathcal{T}_1} d\mathcal{W}^{(l)}(t_l) \right] \times \mathbf{E} \left[\prod_{t_l \in \cup_{j \geq 2} \mathcal{T}_j} d\mathcal{W}^{(l)}(t_l) \right] \right|_{\mathcal{S}_{\mathcal{T}}} \leq Cn^r e^{-\log^2 n}.$$

and this is repeated until we have split of all clusters.

Combining the conclusions 1) and 2), we see that, in total variation

$$\mathbf{E} \left[\prod_{l=1}^r d\mathcal{W}^{(l)}(t_l) \right] = \sum_{\mathcal{T}: d(\mathcal{T}) \geq r/2} \prod_j \mathbf{E} \left[\prod_{t_l \in \mathcal{T}_j} d\mathcal{W}^{(l)}(t_l) \right] + \mathcal{O}(n^{r/2-1/2+\epsilon})$$

where $d(\mathcal{T})$ is the number of clusters. However, since the $d\mathcal{W}^{(l)}(t_l)$ have mean zero, all clusters decompositions \mathcal{T} with singletons vanish on the right hand side. Hence the only leading contributions are those where each cluster consists of a pair, which proves the lemma except for the last statement. That last statement however follows directly from Lemma 2.8. \square

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